Stability of modulated-gravity-induced thermal convection in magnetic fields

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A stability analysis is presented of modulated-gravity-induced thermal convection in a heated fluid layer subject to an applied magnetic field. The nearest correction to the critical Rayleigh number for both single and multiple frequency oscillating-gravity components is obtained by solving the linearized magnetohydrodynamic equations using the small parameter perturbation technique. The correction depends on both the applied magnetic field and the oscillating frequency. In the absence of an applied magnetic field, the correction depends on the Prandtl number only when the exciting frequency is small. However, it asymptotically approaches zero as the frequency increases, with or without the presence of a magnetic field. The heated fluid layer is more stable with gravity modulation than with any type of wall temperature modulation. The difference becomes smaller with decreasing Prandtl number Pr. This finding is of critical importance in that ground-based experiments with appropriate wall temperature modulations may be conducted to simulate the oscillating-gravity effects on the onset of thermal convection in lower-Prandtl-number fluids. For conducting melts considered for microgravity applications, it is possible to apply an external magnetic field to further inhibit the onset of modulated-gravity-induced thermal convection. This effectiveness increases with the Hartmann number Ha. For large Ha, the nearest correction term $R_{02}\sim \mathrm{Ha}^2$ as the magnetic Prandtl number $\mathrm{Pm}\ll 1$. However, $R_{02}\sim \mathrm{Ha}^{4/3}$ for $\mathrm{Ha}\gg 1$ and $\mathrm{Pm}\gg 1$, provided that $\mathrm{Ha}<0.5\pi(\mathrm{Pm}/\mathrm{Pr}^{3/2})$, which is satisfied by a majority of space melt experiments. Thus, under normal laboratory conditions applied magnetic fields are more effective in stabilizing a conducting fluid subject to an oscillating-gravity field than one subject to a constant field. If $\mathrm{Ha}>0.5\pi(\mathrm{Pm}/\mathrm{Pr}^{3/2})$, $R_{02}\sim -\mathrm{Ha}^2$ for $\mathrm{Ha}\gg 1$ and $\mathrm{Pm}\gg 1$ and the magnetic field becomes less effective in stabilizing thermal convection driven by oscillating gravity than that driven by the constant gravity. This is in contrast with the existing studies on thermal convection stability in a magnetic field, which show that marginal stability is independent of Pm and always increases with increasing applied field.

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I. INTRODUCTION

Thermal convection induced by oscillating forces resulting from either oscillating wall temperatures or modulated gravitational forces or a combination of the two has merited attention in the fluids research community for some period of time. Gershuni and Zhukhovitskii [1,2] were among the early investigators to study the instability of thermal convection driven by periodically varying parameters. In their analyses, the Galerkin method was applied to reduce a first-order differential system to an ordinary differential equation with periodic coefficients, which were then solved for critical Rayleigh numbers above which convection sets in. They studied both the classical Bénard problem and convection in a cylinder heated from its bottom. Linear analysis of Bénard convection was also studied by Venezian [3] for a small amplitude modulation of boundary temperatures. Roppo, Davis, and Rosenblat [4] studied the same problem with an oscillating wall temperature condition but their investigations also included weakly nonlinear stability analyses. For finite amplitude analyses, the Galerkin method represents a useful technique, which was used by several authors [2,4–8]. Wadih and Roux [9] also presented a study on the instability of the convection in an infinitely long cylinder with gravity modulation oscillating along the vertical axis. These analyses have all established that the onset of convection is altered under the modulation of constraints. Much recent attention has been on the subharmonic and bifurcation phenomena in fluids under a modulated gravity field [10]. These studies not only are of fundamental value in providing physical insight into the basic behavior of fluids induced by oscillating thermal forces, but also have important implications for designing thermal and fluid systems, e.g., crystal growth from melts in the low-gravity environment of orbiting space vehicles or stations, where gravity changes in both direction and time [9,11,12].

Since molten metals and semiconductor melts are electrically conducting, a magnetic field may be applied to control the thermally induced convective flows in these fluids. Less plausible than gravity, this approach originates from the interaction of the liquid motion with an impressed magnetic field. This interaction gives rise to a Lorentz force that opposes the melt flow. This velocity reduction effect, called magnetic damping, has now been widely used in the materials industry to obtain more homogeneous semiconductors and metal crystals under terrestrial conditions [13]. Because gravity and magnetic fields represent different mechanisms for flow reduction, they may be combined to further suppress the convection in a modulated gravity field [14,15]. Indeed, some research work has been carried out to investigate the possible effects of magnetic fields on oscillating flows. These studies are based on the flow analyses of a conducting fluid in a parallel plate channel subject simultaneously to an oscillating gravity field and a magnetic field. As the channel walls are assumed to be at different temperatures from the fluid and the gravity oscillates perpendicular to the temperature gradient, thermal suction guarantees a time varying convective flow and the system is intrinsically unstable [15].

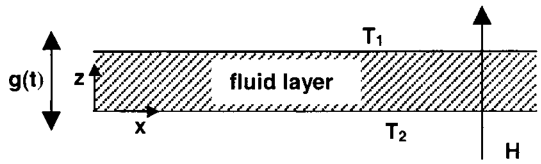

FIG. 1. Schematic of a heated fluid layer subject to the combined action of gravity modulation and an applied magnetic field.

This paper presents a stability analysis of a heated fluid layer subjected to both gravity modulation and an external magnetic field. Both the gravity and the magnetic field are parallel to the temperature gradient. We intend to provide a fundamental understanding of how an applied magnetic field would influence natural convection arising from gravity perturbation. As a first attempt, we present a linear stability analysis of a heated fluid system to explore the effect of magnetic field on oscillating flows. In a subsequent paper, we will present a nonlinear aspect of the problem, which will address the issues of subharmonic and bifurcation phenomena. While the magnetic effect on the convection stability is well understood for fluid flow driven by earth gravity, there seems to have been no work, to the best knowledge of the author, on the magnetic field effect on the stability of thermally driven flows in a modulated-gravity-field environment. Fundamental questions such as how the external field affects the modulated-gravity-driven convection instability and whether or not the understanding gained from studies on the magnetic field effect on constant-gravity-induced flows is pertinent, as well as practical questions such as whether or not a ground-based experiment may be constructed to simulate the low-gravity (or *g*-jitter) effect remain basically unresolved. We intend to answer these questions by solving the magnetohydrodynamic equations using the small parameter perturbation technique, which will form a basis for the subsequent nonlinear analysis.

II. STATEMENT OF THE PROBLEM

Referring to Fig. 1, let us consider a horizontal layer, heated from below, of thickness l in a constant magnetic field and in a modulated gravity environment. A temperature gradient is established in the vertical direction with T_2 at the bottom and T_1 at the top of the fluid layer. The magnetic field is applied perpendicular to the thickness of the fluid and the time dependent gravity is assumed to oscillate along the vertical direction. It has the specific form of

$$g(t)=\mu_g g_0(1+\varepsilon\cos\omega t)+f(t),$$

where $g(t)$ is the gravity perturbation and g_0 the earth gravity constant.

The fluid is assumed to be incompressible and electrically conducting. The density ρ of the fluid is assumed to follow the Bounssinesq approximation, that is,

$$\rho=\rho_0[1-\beta(T-T_r)],$$

where β is the thermal expansion coefficient and ρ_0 the density at the reference temperature T_r. Other thermal and physical properties such as thermal conductivity, electrical conductivity, and viscosity are considered constant. For simplicity, free-free boundary conditions will be prescribed at the top and bottom. These conditions are such that the normal velocities are zero and the tangential stresses are zero at both the top and bottom surfaces. Also, the viscous dissipation is neglected [16]. A dc magnetic field is impressed on the system, and in the present case only the magnetic field in the vertical direction is considered. The objective of this work is to make an assessment of critical conditions for the onset of natural convection with gravity modulation in the presence of an applied dc magnetic field.

III. GOVERNING EQUATIONS

The nondimensionalized form of the magnetohydrodynamic equations describing the fluid flow and heat transfer phenomena in a heated fluid layer subject to both gravity and an applied magnetic field is given by Chandrasekhar [16] and reads

$$\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}=-\nabla P+\mathrm{Ra}(t)\mathrm{Pr}\,T\mathbf{k}+\mathrm{Pr}\,\nabla^2\mathbf{v}+\mathrm{Ha}^2\mathrm{Pm}\,\mathrm{Pr}\,\nabla\times\mathbf{H}\times\mathbf{H}, \tag{1}$$

$$\nabla\cdot\mathbf{v}=0, \tag{2}$$

$$\frac{\partial T}{\partial t}+\mathbf{v}\cdot\nabla T=\nabla^2 T, \tag{3}$$

$$\frac{\partial\mathbf{H}}{\partial t}=\nabla\times(\mathbf{v}\times\mathbf{H})+\mathrm{Pm}\,\nabla^2\mathbf{H}, \tag{4}$$

$$\nabla\cdot\mathbf{H}=0, \tag{5}$$

where $\mathbf{v}$ is the velocity, P the pressure, T the temperature, $\mathbf{H}$ the intensity of the magnetic field, and t the time, nondimensionalized by reference parameters κ/l, $\rho_0\kappa^2/l^2$, T_2-T_1, H_0, and l/κ, respectively. Also in the above equations, Pr is the Prandtl number, Ra the Rayleigh number, Ha the Hartmann number, and Pm/Pr the magnetic Prandtl number. These parameters are defined as follows:

$$\mathrm{Pr}\equiv\frac{\nu}{\kappa},\quad \mathrm{Ra}(t)\equiv\frac{\beta g(t)(T_2-T_1)l^3}{\kappa\nu},$$

$$\mathrm{Ha}^2\equiv\frac{\sigma\mu^2H_0^2l^2}{\rho_0\nu},\quad \mathrm{Pm}\equiv\frac{1}{\sigma\mu\kappa},$$

where ν, κ, β, σ, and μ are the thermophysical properties of the fluid, i.e., the kinematic viscosity, the thermal diffusivity, the volume expansion coefficient, the electrical conductivity, and the magnetic permeability.

The Rayleigh number is a function of time and can be written as

$$\mathrm{Ra}(t)=R(1+\varepsilon\cos\omega t), \tag{6}$$

where R is evaluated using $\mu_g g_0$. For the free-free surface boundaries under present consideration, we have the following simplified boundary conditions:

$$T=1, \quad \frac{\partial u}{\partial z}=\frac{\partial v}{\partial z}=w=0 \quad \text{on} \quad z=0 \tag{7}$$

on the lower plane and

$$T=0, \quad \frac{\partial u}{\partial z}=\frac{\partial v}{\partial z}=w=0 \quad \text{on} \quad z=1 \tag{8}$$

on the upper plane.

In the basic state, the fluid is at rest and thermal conduction prevails. The hydromagnetostatic solutions are as follows:

$$T_0(z,t)=1-z, \quad \mathbf{u}_0=0, \quad P_0(z,t), \quad \mathbf{H}_0=\mathbf{k}.$$

The governing equations for the stability analysis may be derived by considering a small perturbation to the conduction state of the system stated above,

$$\mathbf{v}=\mathbf{u}_0+\mathbf{u}, \quad P=P_0+p, \quad T=T_0+\theta, \quad \mathbf{H}=\mathbf{H}_0+\mathbf{h},$$

where $\mathbf{u}$, p, θ, and $\mathbf{h}$ are the disturbances of velocity, pressure, temperature, and magnetic field.

With these, and taking $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times$ in Eq. (1) to remove the gradient term, one has the equations for the perturbed field variables,

$$\nabla^2\left(\frac{1}{\mathrm{Pr}}\frac{\partial}{\partial t}-\nabla^2\right)\mathbf{u}=\mathrm{Ra}(t)\left(-\frac{\partial}{\partial z}\boldsymbol{\nabla}\theta+\nabla^2\theta\mathbf{k}\right)+\mathrm{Ha}^2\mathrm{Pm}\frac{\partial}{\partial z}\nabla^2\mathbf{h}, \tag{9}$$

$$\left(\frac{\partial}{\partial t}-\nabla^2\right)\theta=W, \tag{10}$$

$$\left(\frac{1}{\mathrm{Pm}}\frac{\partial}{\partial t}-\nabla^2\right)\mathbf{h}=\frac{1}{\mathrm{Pm}}\frac{\partial\mathbf{u}}{\partial z}, \tag{11}$$

where use has been used of $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{h}=-\nabla^2\mathbf{h}$, $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{v}=-\nabla^2\mathbf{v}$, $\boldsymbol{\nabla}\cdot\mathbf{u}=0$, and $\boldsymbol{\nabla}\cdot\mathbf{h}=0$, and also the nonlinear terms have been neglected.

The boundary condition for the perturbed velocity is the same as stated in Eqs. (7) and (8). For the perturbed temperature θ, it is zero at both the top and bottom as the temperature is considered prescribed at these boundaries. The boundary conditions required for the magnetic field for the problem under consideration are such that the field is continuous across the interface between the vacuum and the fluid and that the induced current cannot flow out of the fluid layer, or, specifically, the normal component of the induced current density is zero.

The above linearized equations, along with the boundary conditions, constitute the governing equations for the stability of thermal convection in a magnetic field with gravity modulation. Stability analyses may now be made based on the solution of these equations using the small parameter perturbation method together with Floquet theory [17].

IV. MATHEMATICAL ANALYSIS

In line with the linear stability theory, we analyze the field disturbances in terms of an arbitrary set of normal modes and examine the stability of these modes. Following the procedures given by Chandrasekhar [16], we need only examine the behavior of the vertical component of the velocity field, the temperature field, and the vertical component of the magnetic field. The horizontal components of the velocity and magnetic fields, when needed, may be derived from these solutions. Thus, we can consider the perturbations of these vertical components and the temperature field as two-dimensional waves that take the forms

$$W=w(z,t)\exp[i(a_x x+a_y)],$$

$$\theta=\Theta(z,t)\exp[i(a_x x+a_y y)],$$

$$h_z=\mathcal{H}(z,t)\exp[i(a_x x+a_y y)],$$

where $W=\mathbf{k}\cdot\mathbf{u}$ and $h_z=\mathbf{k}\cdot\mathbf{h}$. With these, Eqs. (9)–(11) may be further simplified, with the result

$$(D^2-a^2)\left(D^2-a^2-\frac{1}{\mathrm{Pr}}\frac{\partial}{\partial t}\right)w+\mathrm{Ha}^2\mathrm{Pm}\,D(D^2-a^2)\mathcal{H}=\mathrm{Ra}(t)a^2\Theta, \tag{12}$$

$$\frac{\partial\Theta}{\partial t}=(D^2-a^2)\Theta+w, \tag{13}$$

$$\frac{\partial\mathcal{H}}{\partial t}=\mathrm{Pm}(D^2-a^2)\mathcal{H}+Dw, \tag{14}$$

where $D=\partial/\partial z$ and $a^2={a_x}^2+{a_y}^2$.

The stability analyses require one to seek the solution of eigenfunctions $(w,\Theta,\mathcal{H})$ and eigenvalues of Ra associated with the above equations for a modulated gravity field that is different from the constant gravity field by a small quantity of order ε. These functions and Ra should be a function of ε and they should be obtained for a given magnetic field or Ha and a frequency ω. Since ε is small for the problem under consideration, we may seek to expand these eigenfunctions and eigenvalues in a series of ε in accordance with the theory of small parameter perturbation [17],

$$\mathrm{Ra}(t)=R(1+\varepsilon\cos\omega t)=(R_{00}+\varepsilon R_{01}+\varepsilon^2 R_{02}+\cdots)\times(1+\varepsilon\cos\omega t), \tag{15}$$

$$w=w_{10}+\varepsilon w_{11}+\varepsilon^2 w_{12}+\cdots, \tag{16}$$

$$\Theta=\Theta_{10}+\varepsilon\Theta_{11}+\varepsilon^2\Theta_{12}+\cdots, \tag{17}$$

$$\mathcal{H}=\mathcal{H}_{10}+\varepsilon\mathcal{H}_{11}+\varepsilon^2\mathcal{H}_{12}+\cdots. \tag{18}$$

The above equations are substituted into Eqs. (12)–(14) and the resulting equations are further cross differentiated to eliminate $\mathcal{H}$. Upon collecting the terms of the same powers of ε, one obtains the set of differential equations

$$L(w_{10})=0,$$

$$\frac{\partial\Theta_{10}}{\partial t}=(D^2-k^2)\Theta_{10}+w_{10}, \tag{19}$$

$$L(w_{11})=-R_{01}a^2\left(\frac{1}{\mathrm{Pm}}\frac{\partial}{\partial t}-(D^2-a^2)\right)w_{10}-R_{00}a^2 \times\left(\frac{\partial}{\partial t}-(D^2-a^2)\right)\left(\frac{1}{\mathrm{Pm}}\frac{\partial}{\partial t}-(D^2-a^2)\right) \times\mathrm{Re}\{e^{-i\omega t}\}\Theta_{10},$$

$$\frac{\partial\Theta_{11}}{\partial t}=(D^2-a^2)\Theta_{11}+w_{11}, \tag{20}$$

$$L(w_{12})=-R_{02}a^2\left(\frac{1}{\mathrm{Pm}}\frac{\partial}{\partial t}-(D^2-a^2)\right)w_{10}-R_{01}a^2 \times\left(\frac{1}{\mathrm{Pm}}\frac{\partial}{\partial t}-(D^2-a^2)\right)w_{11}-R_{01}a^2 \times\left(\frac{\partial}{\partial t}-(D^2-a^2)\right)\left(\frac{1}{\mathrm{Pm}}\frac{\partial}{\partial t}-(D^2-a^2)\right) \times\mathrm{Re}\{e^{-i\omega t}\}\Theta_{10}-R_{00}a^2\left(\frac{\partial}{\partial t}-(D^2-a^2)\right) \times\left(\frac{1}{\mathrm{Pm}}\frac{\partial}{\partial t}-(D^2-a^2)\right)\mathrm{Re}\{e^{-i\omega t}\}\Theta_{11},$$

$$\frac{\partial\Theta_{12}}{\partial t}=(D^2-a^2)\Theta_{12}+w_{12}, \tag{21}$$

where Re means the real value of the quantity in the curly brackets and L is the differential operator defined by

$$L\equiv(D^2-a^2)\left(\frac{\partial}{\partial t}-(D^2-a^2)\right)\left[\left(\frac{1}{\mathrm{Pr}}\frac{\partial}{\partial t}-(D^2-a^2)\right)\left(\frac{1}{\mathrm{Pm}}\frac{\partial}{\partial t}-(D^2-a^2)\right)-\mathrm{Ha}^2D^2\right]+R_{00}a^2\left(\frac{1}{\mathrm{Pm}}\frac{\partial}{\partial t}-(D^2-a^2)\right). \tag{22}$$

The above equations for w_{1i} and $\mathcal{H}_{1i}$ $(i=0,1,2)$ are required to satisfy the boundary conditions given by Eqs. (7) and (8), which may be more specifically stated as follows [16]:

$$w_{1i}=D^2w_{1i}=D^4w_{1i}=0 \quad \text{and} \quad \Theta_{1i}=0 \quad \text{at } z=0,1, \quad i=0,1,2. \tag{23}$$

Equation (19) together with the relevant boundary conditions expressed by Eq. (23) corresponds to the case with $\varepsilon=0$ and describes the standard Bénard convection in a magnetic field. Marginally stable solutions of the eigenfunction of w_{00} and eigenvalue R_{00} for that problem have been obtained by Chandrasekhar [16]. For a fixed value of wave vector a, the eigenfunction is given by

$$w_{00}=\sin\pi z, \tag{24}$$

and the corresponding eigenvalue of R_{00} is computed by

$$R_{00}=\frac{\pi^2+a^2}{a^2}[(\pi^2+a^2)^2+\pi^2\mathrm{Ha}^2]. \tag{25}$$

Note that with $\mathrm{Ha}=0$ or in the absence of a magnetic field the above expression reduces to that for a standard Bénard problem.

To carry out the calculations for higher orders of ε, we need to calculate the perturbed temperature field for the case of $\varepsilon=0$. The solution is obtained by simply substituting w_{00} back into Eq. (19) and solving the resulting equation with the boundary conditions for Θ_{00} as stated in Eq. (23), viz.,

$$\Theta_{00}=\frac{\sin(\pi z)}{\pi^2+a^2}. \tag{26}$$

With the eigenfunctions w_{00} and Θ_{00} substituted into the equation for w_{11}, one has an ordinary differential equation with time periodic coefficients. The mathematical properties and solvability conditions of these types of equation have been extensively studied by Yakubovich & Starzhinskii [17] and by Mishchenko and Rozov [18]. For the equation to have a solution, the theory states that the right hand side of Eq. (20) for w_{11} must be orthogonal to the null space of the operator L. This requires that the time independent (or steady state) part of the right hand side of the equation be orthogonal to its steady state solution or $\sin\pi z$, whence by Eq. (20) R_{01} is zero,

$$R_{01}=\frac{\langle\sin\pi z\overline{F(t)},\sin\pi z\rangle}{\langle\sin\pi z,\sin\pi z\rangle}=0, \tag{27}$$

where the overbar denotes the time average over one cycle of oscillation. The function $\overline{F(t)}$, which is the time dependent part of Eq. (20), is zero as it involves the time averaging of $\cos\omega t$ or $\sin\omega t$. Also, the following definition has been used for the inner product:

$$\langle a(z),b(z)\rangle=\int_0^1 a(z)\cdot b(z)dz. \tag{28}$$

By the same token, one can show that all the odd orders of R, that is, $R_{01},R_{03},R_{05},\ldots$, are also zero because a change of the sign of ε shifts the time origin by a half period but does not change the physical problem [3]. Thus for this type of problem only an even order correction to R exists and the lowest order of correction is R_{02}.

To calculate R_{02}, we need expressions for w_{11} and Θ_{11}. Equation (20) may be solved for w_{11} and Θ_{11} by inverting the operator L. Inspection of the equation suggests that w_{11} and Θ_{11} should take the form of

$$w_{11}=\mathrm{Re}\{Ae^{-i\omega t}\}\sin\pi z \quad \text{and} \quad \Theta_{11}=\mathrm{Re}\{Be^{-i\omega t}\}\sin\pi z. \tag{29}$$

Thus, the following relation may be obtained:

$$L(e^{-i\omega t}\sin\pi z)=L(\omega)e^{-i\omega t}\sin\pi z, \quad (30)$$

where $L(\omega)$ is defined by

$$L(\omega)=\frac{(\mathrm{Pm}+\mathrm{Pr}+1)\omega^2(\pi^2+a^2)^2}{\mathrm{Pm}\,\mathrm{Pr}} +i\omega(\pi^2+a^2)[(1+\mathrm{Pr}^{-1})(\pi^2+a^2)^2 +(1-\mathrm{Pm}^{-1})\pi^2\mathrm{Ha}^2-\omega^2\mathrm{Pr}^{-1}\mathrm{Pm}^{-1}]. \quad (31)$$

Substituting Eqs. (30) and (31) back into Eq. (20) and rearranging the final results, one has the solution for the eigenfunctions w_{11} and Θ_{11} as follows:

$$w_{11}=R_{00}\frac{a^2}{\pi^2+a^2}\times\mathrm{Re}\left\{\frac{(-i\omega+\pi^2+a^2)[i\omega\,\mathrm{Pm}^{-1}-(\pi^2+a^2)]}{L(\omega)}e^{-i\omega t}\right\}\times\sin\pi z, \quad (32)$$

$$\Theta_{11}=R_{00}\frac{a^2}{\pi^2+a^2}\mathrm{Re}\left\{\frac{i\omega\,\mathrm{Pm}^{-1}-(\pi^2+a^2)}{L(\omega)}e^{-i\omega t}\right\}\sin\pi z. \quad (33)$$

These results may be used along with Eq. (22) to determine R_{02}, the lowest-order correction to R. Thus we have for w_{12}

$$L(w_{12})=-a^2R_{02}(\pi^2+a^2)\sin\pi z-a^2R_{00}\left(\frac{\partial}{\partial t}-(D^2-a^2)\right)\times\left(\frac{1}{P_m}\frac{\partial}{\partial t}-(D^2-a^2)\right)\mathrm{Re}\{e^{-i\omega t}\}\Theta_{11}. \quad (34)$$

By the solvability condition for Eq. (34), the steady part of its right-hand side needs to be made orthogonal to $\sin\pi z$, or the steady state solution, whence we have the result for the eigenvalue of R_{02}:

$$R_{02}=-\frac{2R_{00}}{(\pi^2+a^2)}\left\langle\overline{\left(\frac{\partial}{\partial t}-(D^2-a^2)\right)\left(\frac{1}{\mathrm{Pm}}\frac{\partial}{\partial t}-(D^2-a^2)\right)\mathrm{Re}\{e^{-i\omega t}\}\Theta_{11}},\sin\pi z\right\rangle. \quad (35)$$

Carrying out the appropriate temporal and spatial operations, one has the final solution for R_{02},

$$R_{02}=\frac{R_{00}^2a^2}{2}\left(\mathrm{Re}\left\{\frac{(\pi^2+a^2)\mathrm{Pm}-i\omega}{\mathrm{Pm}\,L(\omega)}\right\}\right). \quad (36)$$

We can now determine the smallest wave number of the disturbance that will amplify thermal convection in the fluid layer when the Rayleigh number reaches a threshold value. From Eqs. (25) and (36), the wave number is a function of the applied magnetic field. The minimum value of wave number at which the critical Rayleigh number exists can be derived by taking the derivative of R and setting it to zero,

$$\partial R/\partial a_c=\partial R_{00}/\partial a_c+\varepsilon^2\partial R_{02}/\partial a_c+\cdots=0. \quad (37)$$

In addition, the wave number can also be expanded in powers of ε, or

$$a_c=a_0+\varepsilon a_1+\varepsilon^2a_2+\cdots. \quad (38)$$

With these, one can show that to the leading order a_c is calculated by

$$\frac{\partial R_{00}}{\partial a_c}=0, \quad (39)$$

which gives the following equation for determining the critical wave number a_c:

$$2a_c^6+3\pi^2a_c^4=\pi^6+\pi^4\mathrm{Ha}^2. \quad (40)$$

This is the same as that for unmodulated Bénard convection in a magnetic field [16]. Thus, to the order of ε^2, the critical Rayleigh number is calculated by evaluating R_{00} and R_{02} at $a=a_c$. In his studies on Bénard convection with a modulated temperature gradient, Venezian [3] showed that a_1 is zero and a_2 needs to be considered only when a higher-order approximation, say R_{04}, is evaluated. It is straightforward to show that this conclusion applies to the present problem as well. In passing, we note that with the absence of an applied magnetic field, Eq. (40) gives $a_c^2=\pi^2/2$, the same as for the pure Bénard problem, as expected. However, unlike in the pure Bénard problem for which a_c is a constant, a_c becomes dependent on the magnetic field applied when present. In fact, $a_c\to\pi^{2/3}\mathrm{Ha}^{1/3}/2^{1/6}$ and $R_{00}\to\pi^2\mathrm{Ha}^2$ when $\mathrm{Ha}\to\infty$, which implies that, as the applied magnetic field increases, the wavelength of the plane disturbances (i.e., $2\pi l/a_c$) at marginal stability becomes increasingly smaller and so does the size of the cellular structure.

V. LIMITING BEHAVIOR OF R_{02}/R_{00}

Before embarking upon a discussion of the results, let us examine some of the limiting behavior of the solutions. In that regard, we are particularly interested in the ratio R_{02}/R_{00}. From the solutions for R_{00} and R_{02}, the ratio depends on the exciting frequency and amplitudes, the magnetic field, and the thermophysical properties of the fluids.

A. The case when Ha→0

This case corresponds to Bénard convection driven by an oscillating gravity field without the presence of a magnetic field. Gresho and Sani [5] studied the stability problem of a fluid layer with gravity modulation using the Gerlakin method and Hill's theory. Our present solutions may be used to make an assessment of the problem of a qualitative nature. This can be done by setting the Hartmann number to zero in Eqs. (25) and (35), which gives the result

$$\frac{R_{02}}{R_{00}}=\frac{(\pi^2+a^2)^2\mathrm{Pr}}{2[(\pi^2+a^2)^2(\mathrm{Pr}+1)^2+\omega^2]}. \tag{41}$$

This equation suggests that R_{02} is always positive and the fluid becomes more stable with a modulated gravity field than with an unmodulated one. The stability of the fluid layer depends on the Prandtl number of the liquid, the excitation frequency, and the other parameters associated with R_{00}. The maximum stabilizing effect occurs at $\omega=0$. This conclusion is consistent with Venezian's result [3] on thermal convection with an oscillating wall boundary condition. For both the present case and Venezian's problem, R_{02}/R_{00} approaches an asymptotic value [i.e., $0.5\,\mathrm{Pr}/(1+\mathrm{Pr})^2$] at a rate of ω^2 as $\omega\to 0$ when the wall temperatures are oscillated out of phase, while the stabilizing effect diminishes to zero at a rate of ω^{-2} as $\omega\to\infty$. This behavior is also consistent with the analyses of Wadih and Roux [9], who studied thermal convection in an infinitely long cylinder with gravity modulation along the axis of the cylinder. In fact, with their R_{00}^c substituted for $(\pi^2+a^2)^2$, the above equation becomes identical to the expression for the ratio of R_{02}/R_{00} for an infinitely long cylinder with a vertical gravity modulation [see Eq. (8.11) in [9]]. Equation (41), together with the results obtained by other investigators [3,6,9], seems to suggest that the near order correction to the critical Rayleigh number is a constant at $\omega=0$, and the constant is given by $R_{02}=27\pi^4/8\,\mathrm{Pr}(1+\mathrm{Pr}^{-1})^2$, which is a function of Prandtl number and is independent of the type of geometry and the mechanism of periodic excitation (whether bottom wall temperature or antisymmetric wall temperature or gravity modulation). An exception to this is when both the top and bottom wall temperatures are modulated in phase, for which case $R_{02}=0$ at $\omega=0$ (see also case *a* in Fig. 3 below).

B. The case when Pm→0 or Pm→∞

The parameter Pm is a measure of the ratio of diffusion of magnetic field over diffusion of heat in a medium. For materials that transport heat much faster than magnetic field, Pm→0 and thus we have the relationship,

$$\frac{R_{02}}{R_{00}}\to\frac{(\omega^2+\mathrm{Ha}^2\pi^2\mathrm{Pr})[(\pi^2+a^2)^2+\mathrm{Ha}^2\pi^2]\mathrm{Pr}}{2[\omega^2(\pi^2+a^2)^2(\mathrm{Pr}+1)^2+(\omega^2+\mathrm{Ha}^2\pi^2\mathrm{Pr})]^2}\quad \text{as}\ \ \mathrm{Pm}\to 0.$$

It is remarked that the above result should apply to the monotonic stability branch as overstability may occur when $\mathrm{Pm}\to 0$ or $1/\sigma\mu\ll\kappa$ [18].

On the other hand, for materials in which the magnetic field diffuses much faster than heat, $\mathrm{Pm}\to\infty$ or $1/\sigma\mu\gg\kappa$ (most metal and semiconductor melts fall into this category and have a value of $\mathrm{Pm}\sim 1\times 10^3$ or $\mathrm{Pm/Pr}\sim 1\times 10^5$), the following relationship is obtained:

$$\frac{R_{02}}{R_{00}}\to\frac{(\pi^2+a^2)^2[(\pi^2+a^2)^2+\mathrm{Ha}^2\pi^2]\mathrm{Pr}}{2\{\omega^2(\pi^2+a^2)^2+[(\pi^2+a^2)^2(\mathrm{Pr}+1)+\mathrm{Ha}^2\pi^2]^2\}}\quad \text{as}\ \ \mathrm{Pm}\to\infty.$$

For both cases, R_{02} is always positive and thus an applied magnetic field increases the critical Rayleigh number, thereby inhibiting thermal convection. Indeed, because R_{02} is positive, an applied magnetic field becomes even more effective in damping out disturbance in a fluid subjective to a modulated gravity field than one subject to a constant gravity field. Both relations reduce to Eq. (41) when Ha→0, as is expected. For a fixed magnetic field, the effect of the frequency on the stability of the fluid layer differs for the two cases. The difference, however, diminishes as ω increases. In fact, both relations show that $R_{02}/R_{00}\sim\omega^{-2}$ as $\omega\to\infty$ for a fixed Ha. However, the stability behavior with an increase in the applied magnetic field differs for these two limiting situations. In the case of Pm→0, the ratio R_{02}/R_{00} approaches a constant as Ha→∞, suggesting that for these types of material R_{02} increases at a rate of Ha^2, i.e., the same as R_{00}. For the case of Pm→∞, on the other hand, $R_{02}/R_{00}\sim\mathrm{Ha}^{-2/3}$, or R_{02} increases at rate of $\mathrm{Ha}^{4/3}$, as Ha→∞. This is in sharp contrast with the conclusion drawn from a stability study of fluid subject to a constant gravity in a magnetic field, which states that marginal stability is independent of the parameter Pm [16].

It should be remarked here that the above discussion for the case of Pm→∞ is conditional on $\mathrm{Ha/Pm}\gg 1$. However, detailed analysis shows that the conclusion holds true so long as $\mathrm{Ha}<0.5(\mathrm{Pm/Pr})^{3/2}$. Thus, for typical electrically conducting melt experiments under consideration for space applications, Ha is in general less than 1000 while $0.5\pi(\mathrm{Pm/Pr})^{3/2}\sim 5\times 10^7$. Clearly the condition is well satisfied.

The analyses further show that, if $\mathrm{Ha}>0.5\pi(\mathrm{Pm/Pr})^{3/2}$, $R_{02}\sim-\mathrm{Ha}^2$ as $(\mathrm{Ha,Pm})\to\infty$ and thus an applied magnetic field becomes less effective in stabilizing the fluid in a modulated than in a constant gravity field. Moreover, $R_{02}\sim-\mathrm{Ha}^2/(\mathrm{Pm}-1)$ as Ha→∞ for a finite value of Pm ($\neq 1$).

VI. RESULTS AND DISCUSSION

Some numerical results are provided below to further illustrate the onset of thermal convection with oscillating gravity and the effects of applied magnetic fields. Figure 2 plots the results for R_{02}/R_{00} against ω for free convection by oscillating gravity forces without an applied magnetic field, along with the results by Venezian [3] for convection by wall temperatures modulated out of phase [see Eq. (45) in his paper] for four values of Pr, namely, Pr=0.01, 0.1, 1, and 10.

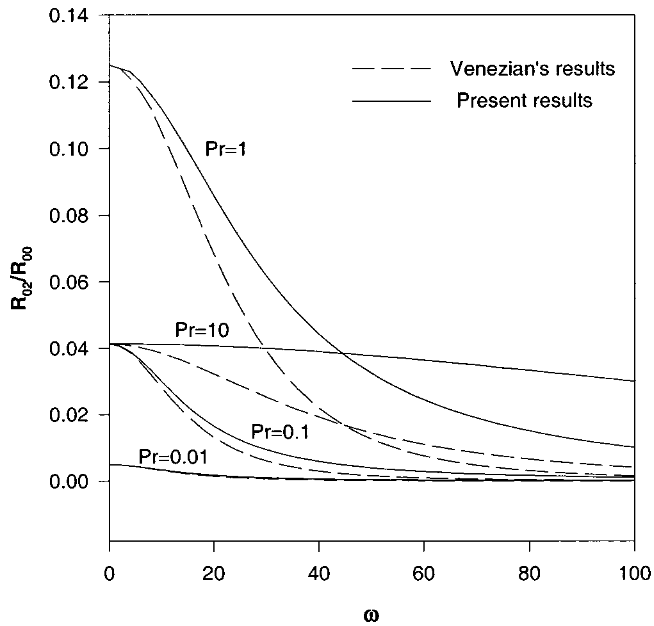

FIG. 2. Comparison of R_{02}/R_{00} for gravity modulation and wall temperature modulation in the absence of an external magnetic field.

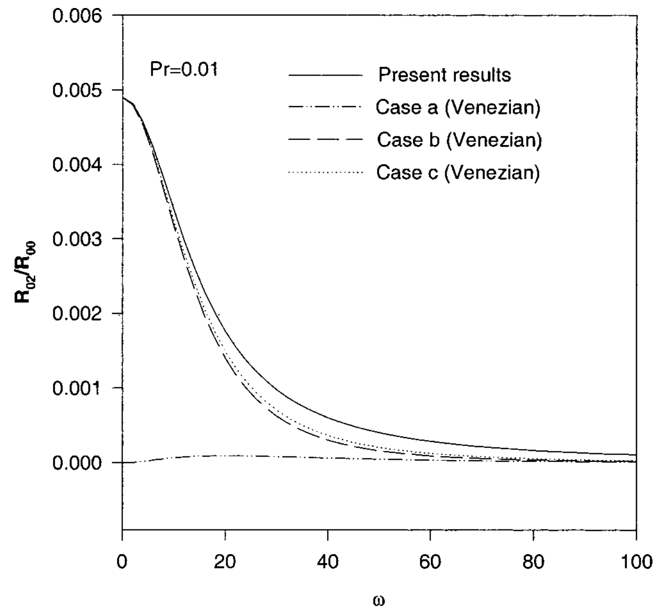

FIG. 3. Comparison of R_{02}/R_{00} for gravity modulation and three different wall temperature modulations for Pr=0.01 without an applied magnetic field.

It is apparent that the results for the two different modulations match exactly at the limit $\omega=0$ for all values of Pr, as expected. With an increase in ω, the ratio of R_{02}/R_{00} decreases for both gravity and wall temperature modulations. However, R_{02}/R_{00} decreases much faster with wall temperature modulation than with gravity modulation for the entire range of Pr. The same behavior was also reported by Wadith and Roux [9] for an infinitely long cylinder with gravity modulation along the axis of symmetry.

Inspection of Fig. 2 also indicates that as Pr decreases the difference for the onset of convection between the wall temperature modulation and gravity modulation becomes smaller. In particular, at Pr=0.01, the two curves for different modulations almost overlap. To better resolve the difference, the present results are plotted in Fig. 3 along with those obtained for three different wall temperature oscillating mechanisms, where cases *a* and *b* correspond to symmetric and antisymmetric wall temperature modulations, respectively, and case *c* is for the bottom wall temperature modulation only. Clearly, case *a* is much different; but cases *b* and *c* are very closely correlated to the present results, with a maximum difference of <2% for a frequency of less than 10. Perhaps one of the important implications of these results is that a ground-based experimental system that employs wall temperature modulations, either symmetric or antisymmetric, may be designed to simulate reasonably well the effects of gravity modulations, for the purpose of studying the onset of thermal convection of a heated fluid layer with these modulations.

The magnetic field effects on the thermal convection are depicted in Fig. 4 for liquids with Pr=0.01, which covers most electrically conducting fluids. For a majority of microgravity experiment systems considered for metal and semiconductor melts, the Hartmann number is in the range of 50 to 100 and a Hartmann number of 1000 is considered an upper limit. For example, for a Ga-doped germanium single crystal growth system of 1 cm in dimension, Ha=1000 would correspond to a magnetic field of 2.2 T [15]. The results in Fig. 4 clearly indicate that an applied magnetic field helps to stabilize the thermal convection subject to gravity modulations. In fact, the fluid becomes more stable as the magnetic field strength (or the Hartmann number Ha) increases. This behavior is further exhibited in Fig. 5, where R_{02}/R_{00} is plotted against Ha with ω as an additional parameter.

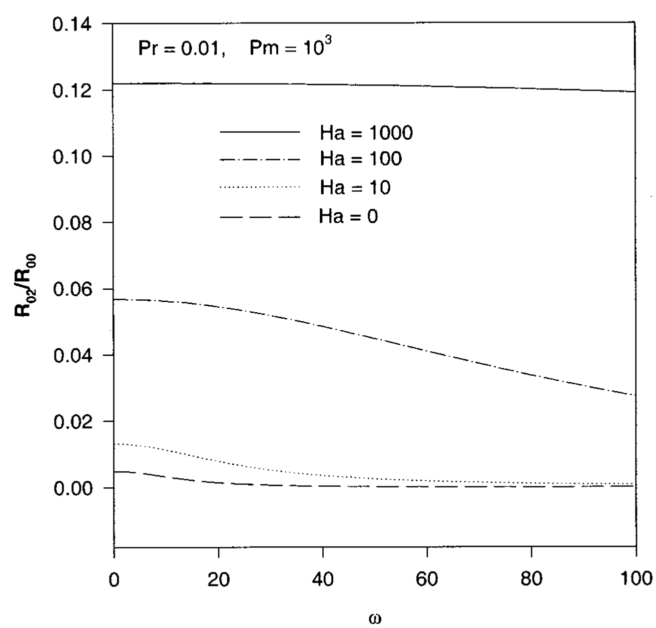

FIG. 4. Dependency of R_{02}/R_{00} on frequency of *g*-jitter modulation in the presence of an applied magnetic field for Pr=0.01.

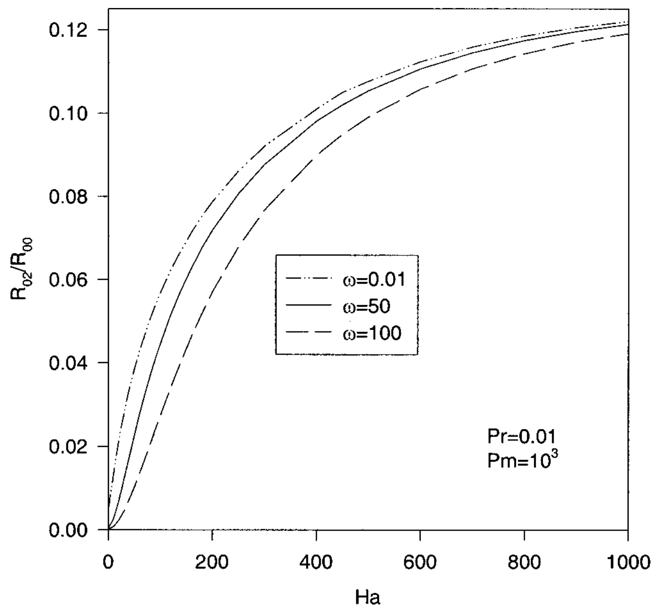

FIG. 5. Dependency of R_{02}/R_{00} on the strength of an applied magnetic field for Pr = 0.01.

VII. EXTENSION TO MULTIFREQUENCY MODULATION

In space vehicles, gravity in tandem with microgravity is random and is caused by various sources including crew movement and on-board machine operations. In many cases, a single component is not sufficient to present the gravity perturbation and these g-jitter data may be represented by a synthesized Fourier series, each term of which involves a time harmonic function with a distinct frequency,

$$g(t)=\mu_0 g_0\left(1+\sum_{m=1}^{M}\varepsilon_m\cos\omega_m t\right)=\mu_0 g_0(1+\mathbf{E}^T\cos\boldsymbol{\omega}t), \tag{42}$$

where $\mathbf{E}$ and $\cos\boldsymbol{\omega}t$ are two vectors,

$$\mathbf{E}=\{\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots\}^T \quad \text{and} \quad \cos\boldsymbol{\omega}t =\{\cos\omega_1 t,\cos\omega_2 t,\cos\omega_3 t,\ldots\}^T. \tag{43}$$

The linear analyses presented in the previous sections for a single component can be adapted to study these types of multifrequency component modulation. In particular, Eqs. (19)–(23) will be used to describe each individual component with distinct frequency and amplitude. Now, similar procedures may be extended to solve for multiple frequency g-jitter problems. Following the same line of argument as stated before, R and W are expanded in terms of $\mathbf{E}$,

$$R=R_0+\mathbf{E}^T\cdot\mathbf{R}_{01}+\mathbf{E}^{2T}\cdot\mathbf{R}_{02}+\cdots, \tag{44}$$

$$W=W_0+\mathbf{E}^T\cdot\mathbf{W}_{01}+\mathbf{E}^{2T}\cdot\mathbf{W}_{02}+\cdots, \tag{45}$$

where $\mathbf{E}^2$, $\mathbf{R}_{01}$, $\mathbf{W}_{01}$, $\mathbf{R}_{02}$, and $\mathbf{W}_{02}$ are also vectors and are defined by

$$\mathbf{E}^2=\{\varepsilon_1\varepsilon_1,\varepsilon_1\varepsilon_2,\varepsilon_1\varepsilon_3,\ldots,\varepsilon_i\varepsilon_j,\ldots\}^T,$$

$$\mathbf{R}_{01}=\{R_{01}^1,R_{01}^2,R_{01}^3,\ldots\}^T,$$

$$\mathbf{W}_{01}=\{W_{01}^1,W_{01}^2,W_{01}^3,\ldots\}^T,$$

$$\mathbf{R}_{02}=\{R_{02}^{11},R_{02}^{12},R_{02}^{13},\ldots,R_{02}^{ij},\ldots\}^T,$$

$$\mathbf{W}_{02}=\{W_{02}^{11},W_{02}^{12},W_{02}^{13},\ldots,W_{02}^{ij},\ldots\}^T.$$

Θ and $\mathcal{H}$ can be expanded in a similar fashion. Substituting these relations into Eqs. (12)–(14) and separating the same order terms as was done before for the single frequency case, we have a system of differential equations, similar to Eqs. (19)–(23), for the expanded variables.

This set of equations can be solved in a fashion parallel to the system defined by Eqs. (19)–(23). In particular, we note that $\mathbf{R}_{02}$, and similarly other odd term corrections, vanish by the same reasoning based on the solvability condition for Eq. (20). Thus, the first-order functions such as $\mathbf{W}_1$ can be determined and should have expressions similar to those in Eq. (29). With these results and also the solvability requirement that the steady part of the right hand side of Eq. (42) be orthogonal to $\mathbf{W}_0,\mathbf{R}_{02}$ can be determined. Writing the solution in component form, we have for R_{02}^{mn},

$$R_{02}^{mn}=-\frac{R_{00}}{(\pi^2+a^2)}\left\langle\overline{\left(\frac{\partial}{\partial t}-(D^2-a^2)\right)\left(\frac{1}{Pm}\frac{\partial}{\partial t}-(D^2-a^2)\right)\mathcal{R}\{e^{-i\omega_m t}\}\Theta_{11}^n},\sin\pi\right\rangle, \tag{46}$$

where

$$\Theta_{11}^n=R_{00}\frac{a^2}{\pi^2+a^2}\mathrm{Re}\left\{\frac{i\omega_n\mathrm{Pm}^{-1}-(\pi^2+a^2)}{L(\omega_n)}e^{-i\omega_n t}\right\}\sin\pi z. \tag{47}$$

In particular if $\omega_n=n\omega_1$ where n is an integer and ω_1 is the reference frequency, the cross product terms disappear and we have a simplified equation for R_{02}^{mn},

$$R_{02}^{mn}=\begin{cases}\dfrac{R_{00}^2a^2}{2}\mathrm{Re}\left\{\dfrac{(\pi^2+a^2)\mathrm{Pm}-i\omega_n}{\mathrm{Pm}\,L(\omega_n)}\right\} & \text{for } m=n\\ 0 & \text{for } m\neq n,\end{cases} \tag{48}$$

which obviously is very similar to that obtained for a single frequency g-jitter component.

VIII. CONCLUDING REMARKS

This paper has presented a stability study of modulated-gravity-induced thermal convection subject to an applied magnetic field. The analysis is based on solution of the linearized magnetohydrodynamic equations using the small parameter perturbation technique. The nearest correction to the critical Rayleigh number above which thermal convection sets in was obtained for both single and multiple frequency modulations. The nearest correction term R_{02} is found to be a function of both applied magnetic field and gravity modulation frequency. The term asymptotically approaches a constant at a rate of ω^2 for $\omega \ll 1$, while it goes to zero at a rate of ω^{-2} for $\omega \gg 1$. This holds true with or without the presence of a magnetic field. The heated fluid layer is more stable with gravity modulation than with wall temperature modulations. The difference, however, becomes smaller with decreasing Prandtl number. For metals and semiconductor melts ($\mathrm{Pr}=0.01$), the difference becomes reasonably small so that the modulated gravity effects on flow instability may be simulated with appropriately designed wall temperature modulations. For conducting melts, modulated-gravity-induced thermal convection can be suppressed by applying an external magnetic field. In fact, the magnetic field is more effective in stabilizing fluids subject to a modulated gravity field than fluids subject to a constant one. The correction term R_{02} increases at a rate of Ha^2 for $\mathrm{Pm} \ll 1$, but at a rate of $\mathrm{Ha}^{4/3}$ for $\mathrm{Pm} \gg 1$ in fluids in a modulated gravity field if $\mathrm{Ha} < 0.5\pi(\mathrm{Pm}/\mathrm{Pr})^{3/2}$, which is satisfied by most space melt experiments under consideration. For the case of $\mathrm{Ha} > 0.5\pi(\mathrm{Pm}/\mathrm{Pr})^{3/2}$, R_{02} *decreases* at a rate of Ha^2 for $\mathrm{Pm} \gg 1$, indicating that an increase in applied magnetic field strength may destabilize the fluids when subjected to an oscillating gravity. This is in sharp contrast with our existing knowledge that the marginal stability of a magnetic field acting on fluids subjected to constant gravity always increases with Ha^2 but is independent of Pm.

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